COMPACT TOPOLOGICAL LATTICES

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TABLE OF CONTENTS

Page

ACKNOWLEDGMENTS ii

Section

I. INTRODUCTION 1

II. TOTAL DISCONNECTEDNESS IN A COMPACT TOPOLOGICAL LATTICE 7

III. TOPOLOGICAL LATTICES AND INTRINSIC TOPOLOGIES 21

IV. COMPACT AND CONNECTED DISTRIBUTIVE TOPOLOGICAL LATTICES 36

BIBLIOGRAPHY 54

BIOGRAPHICAL SKETCH 56

SECTION I

INTRODUCTION

A topological lattice is one of the topological algebras and is defined as a Hausdorff space endowed with a pair of continuous lattice operations in a manner similar to that of topological semigroups, topological groups, topological rings and Banach lattices.

In this paper the main purpose is to study the relationships between the topological properties and the algebraic properties of a topological lattice. In other words, the topological properties of the system as reflected in its algebraic structure (and vice versa) is being investigated.

During the past decade, topological lattices (more generally, partially ordered spaces) have been the object of intense study. Some authors, for example, Anderson [1, 3, 13, 19] and Wallace [4, 10], have considered the characterization of locally compact topological lattices using the codimension of Cohen [20]. These authors have considered the relationship between the number of center elements [2] and the codimension of a compact and connected topological lattice. Particularly interesting is Anderson's result which asserts that every locally compact and connected topological lattice of codimension less than or equal to one is a chain [1].

Other investigators, for example, Birkhoff [2], Frink [7], Rennie [23], and Ward [25], have published results on the interval and order topologies defined on partially ordered sets and lattices. Although it turns out that these induced topologies do not always yield topological lattices, it is nevertheless of interest to investigate conditions under which a given intrinsic topology does yield a topological lattice. As we show in Section III, for example, a lattice-ordered group with the chain condition (defined in [2]) is a topological lattice in the interval topology if and only if it is a chain (i.e., it is a cyclic group). Recently, however, C. Holland [17] has given an example of a lattice-ordered group which is both a topological lattice and a topological group in the interval topology. The question is still open as to what conditions are necessary and sufficient for a lattice-ordered group to be a topological lattice or to be a topological group.

Section II of this paper is concerned with, first, the total disconnectedness of a compact Boolean topological lattice. In [5], Kaplansky proved that a compact semi-simple topological ring is iseomorphic (isomorphic and homeomorphic) with a Cartesian direct sum of finite simple rings; this implies that any compact Boolean topological lattice is totally disconnected.

In the proof of this theorem, Kaplansky used the character group in the sense of a topological group (the so-called duality theorem). Professor A. D. Wallace has suggested the possibility of a proof of the latter theorem which does not involve the use of the duality theorem. In this connection we prove the following:

If L is a non-degenerate, compact, connected, distributive topological lattice, and if every non-unit element has a non-zero disjoint element in L, then the codimension of L is infinite.

In the proof of this theorem, the main tools are the results of Anderson [1, 3], Dyer and Shields [14], and Wallace [10]. From the above theorem, it follows that any compact Boolean topological lattice L of finite codimension is always totally disconnected.

If one is allowed to use the duality theorem, then one can replace the compactness hypothesis in the theorem by suitable local properties of L as follows: If L is a locally compact and locally convex [6] Boolean topological lattice, then L is totally disconnected.

In the remainder of this section we will exhibit a sufficient condition for a compact and complemented modular topological lattice to be totally disconnected. To do so, we show that any two projective intervals [2] in a modular topological lattice are topologically equivalent. Using several lemmas and the results of Anderson [1], we have the theorem: If L is a compact and complemented modular topological lattice, and if (*) $a \wedge L \neq F(a \wedge L)$ for all non-zero elements a of L and dually, then L is totally disconnected where F denotes the boundary operator. The condition (*) in this theorem may be too strong, since the author has not yet been able to demonstrate the necessity of the condition in such a topological lattice.

In Section III we investigate some of the relationships between topological lattices and their intrinsic topologies. The main results are: If a lattice L is Hausdorff in the interval topology, denoted by $\mathcal{I}$, and if $(L, \mathcal{T})$ is a topological lattice in the topology $\mathcal{T}$, then $(L, \mathcal{T})$ is compact if and only if $\mathcal{T} = \mathcal{I}$ and L is a complete lattice. It is also shown that if a lattice L is compact in the order topology, denoted by $\mathcal{O}$, and if $(L, \mathcal{T})$ is a compact topological lattice, then $\mathcal{T} = \mathcal{O}$.

In [15], Floyd has shown that a complete Boolean algebra is not necessarily Hausdorff in the order topology. In this connection, we show that if L is a complete Boolean algebra and if either the meet lattice operation or the join lattice operation of L is continuous in the order topology $\mathcal{O}$, then L is Hausdorff in $\mathcal{O}$ and, hence, L is a topological lattice in $\mathcal{O}$.

In addition, using the results of Birkhoff [2] and Baer [9], we have shown, as stated before, that among the non-degenerate lattice-ordered groups satisfying the chain condition, the only topological lattices are chains. It is also shown that if L is a complete lattice-ordered group, then L is discrete in its order topology if and only if L has the chain condition.

In Section IV another method of proving Wallace's conjecture, which appeared in [10] and [3], will be demonstrated and the representation theorems of a distributive topological lattice will also be considered. In fact, a topological lattice is characterized as follows:

A topological lattice L with 0 and 1 is iseomorphic with an n-cell if and only if L is distributive and contains n independent elements x_i [2] over 0, whose union is 1, such that each $x_i \wedge L$ is a separable connected locally convex chain.

This theorem generalizes somewhat a result of Anderson [13].

In addition, several structure theorems of compact topological lattices are proved. Among them is the result that if L is a locally compact and chain-wise connected distributive topological lattice with 0 and 1, the following are equivalent:

(i) Ca(L) = cd(L) = n, where Ca(L) is the number of all atoms in the center of L,

(ii) Ca(L) = n and for each atom x_i of the center in L, $x_i \wedge L$ is a chain, and

(iii) L is iseomorphic with n Cartesian products of compact connected chains.

After introducing the definition of the finite ordinal sum of topological lattices with 0 and 1, it can be shown that if L is a compact connected distributive topological lattice, then L is iseomorphic with a finite ordinal sum of finite Cartesian products of compact and connected chains if and only if there exists a finite chain: $0 = a_o < a_1 < \ldots < a_n = 1$ of cut points $a_1, \ldots, a_{n-1}$ of L such that $\mathrm{Card}(\mathrm{Cen}[a_{i-1}, a_i]) \geq \mathrm{Card}(\mathrm{Cen}[c,d])$ for any pair of c and d with $a_{i-1} \leq c \leq d \leq a_i$, $i = 1, 2, \ldots, n$.

Finally, it is also shown that if L is a locally compact and connected topological lattice and if L is a union of a countable family of closed connected chains, then L is a chain.

SECTION II

TOTAL DISCONNECTEDNESS IN A COMPACT TOPOLOGICAL LATTICE

We review some definition, notation, and preliminary results which will be needed in the sequel.

The lattice operations of join and meet are designated by $\vee$ and $\wedge$, respectively. We use the symbols 0 and 1 to denote the zero and unit in a lattice or in a Boolean algebra. Set operations are indicated by rounded symbols: $\cap$, $\cup$ and $\subseteq$ stand for intersection, union and inclusion, respectively. The empty set is denoted by $\Box$.

We define a topological lattice to be a pair $(L, \mathcal{T})$ where L is a lattice with two operation $\wedge$ and $\vee$, and is a Hausdorff topological space under the topology $\mathcal{T}$ such that

$$\vee : L \times L \rightarrow L \text{ and } \wedge : L \times L \rightarrow L$$

are continuous mappings. We agree to write $\vee(x,y) = x \vee y$ and $\wedge(x,y) = x \wedge y$ for all x and y in L.

For a pair of subsets A and B of L, we shall use $A \wedge B$ and $A \vee B$ to denote the sets $\{a \wedge b | a \in A \text{ and } b \in B\}$ and $\{a \vee b | a \in A \text{ and } b \in B\}$, respectively. For a subset A of L, A^*, A^o and $F(A)$ denote the closure, the interior, and the boundary of A, respectively. The subset A of a topological space L is a retract of L if there is a continuous mapping f of L into A such that $f|A$ (the restriction of f on A) is the identity.

(2.1) Lemma. If L is a topological lattice, then $a \wedge L$ and $a \vee L$ are closed subsets in L for all $a \in L$.

Proof. We show that $a \wedge L$ is closed. Define a mapping f of L into $a \wedge L$ by $f(x) = a \wedge x$. By the definition of a topological lattice, f is continuous. Since $x \in a \wedge L$ implies that $f(x) = a \wedge x = x$, we have that $f|a \wedge L$ is the identity. So $a \wedge L$ is a retract of L. It is well known that every retract of a Hausdorff space is closed. Therefore, $a \wedge L$ is closed in L for all $a \in L$. Dually, we have that $a \vee L$ is closed.

A more general form of the above lemma appears in [1]: if L is a topological lattice and if A is a compact subset of L, then $A \wedge L$ and $A \vee L$ are closed.

If L is a topological lattice with zero element 0 and if x and y are such elements of L that $x \wedge y = 0$, then x and y are termed disjoint elements.

If L is a topological lattice, we say that $A \subset L$ is convex provided $x \leq y \leq z$ implies $y \in A$ if x and z are in A. Anderson [1] has shown that A is convex if and only if $A = (A \wedge L) \cap (A \vee L)$, where $(A \wedge L) \cap (A \vee L)$ is denoted by C(A), and has shown that

(i) C(A) is the smallest convex set containing A.

(ii) If A is compact, then C(A) is closed.

(iii) If A is open, then C(A) is open.

(iv) If A is convex, then A^{o} is convex.

(v) If A is connected, then $C(A)$ is connected.

Let L be a lattice. The breadth [2] of L, denoted by $Br(L)$, is the smallest integer n such that any finite subset F of L has a subset F' of, at most, n elements such that $\mathrm{Inf}\ F = \mathrm{Inf}\ F'$.

$H^{n}(X)$ will denote the n-dimensional Alexander-Kolmogroff cohomology group of the topological space X with coefficients in some non-trivial additive abelian group G [24, 25]. We define $X \in F^{-1}(G)$ if and only if $X = \square$, and $X \in D^{n}(G)$ for $n \geq 0$ if and only if X is a locally compact Hausdorff space such that if $A \subset X$ is compact, if $B = B^{*} \subset A$, and if the inclusion mapping $i : B \subset A$, then the induced mapping $i^{*} : H^{n}(A) \to H^{n}(B)$ is an epimorphism. The greatest lower bound of $\{n \mid X \in D^{n}(G)\}$ is called the codimension of the space X, denoted by $cd(X)$.

Now we shall prove the following theorem:

(2.2) Theorem. If L is a non-degenerate compact and connected distributive topological lattice, and if every non-unit element has a non-zero disjoint element in L, then the codimension of L is infinite.

Before the proof of the theorem, we review some theorems which will be needed in the proof.

The following theorem is due to E. Dyer and A. Shields [14]:

(2.3) Theorem. If L is a compact distributive topological lattice of finite breadth and U is a neighborhood of a point $x \in L$, then there is a convex set $C_{y,z} = \{a|y \leq a \leq z\}$ that is a neighborhood of x which lies in U.

We also use the following results of L. Anderson [1, 3].

(2.4) Lemma. If L is a connected topological lattice and if $a \in L$, then

$$a \wedge [L\backslash(a \wedge L)] \subset F(a \wedge L), \text{ and dually.}$$

(2.5) Corollary. If L is a locally compact, chain-wise connected distributive topological lattice, then $Br(L) \leq cd(L)$, where if L is a topological lattice, then L is chain-wise connected if for each pair of elements x and y in L with $x \leq y$ there is a closed connected chain from x to y. It is well known that a compact connected topological lattice is chain-wise connected.

A. D. Wallace [10] has shown:

(2.6) Theorem. If L is a compact topological lattice, then L has 0 and 1.

Proof of Theorem (2.2). Suppose that cd(L) is finite. Since L is a compact connected topological lattice, L is locally compact and chain-wise connected. Therefore, by Corollary (2.5) we have $Br(L) \leq cd(L)$, which implies Br(L) is also finite. According to (2.6), L has a zero element 0 and a unit 1, and we can find a neighborhood U of 0 such that $1 \notin U$. Now by (2.3) there exists a closed interval $C_{y,z} = \{x|y \leq x \leq z\}$ that is a neighborhood of 0 which lies in U. However, $0 \in C_{y,z}$ implies $y = 0$. Therefore, $C_{y,z} = C_{o,z} = z \wedge L$.

Since $z \wedge L$ is a neighborhood of 0, there exists an open neighborhood V of 0 such that $0 \in V \subset C_{o,z} = z \wedge L \subset U$. Suppose $z = 0$. We then have $C_{o,z} = \{0\} = V$. Therefore, $\{0\}$ is an open subset, i.e., $\{0\}$ is closed and open in L, which is a contradiction because L is non-degenerate and connected. Hence, $z > 0$ and $z \neq 1$ since $z \in U$ and $1 \notin U$.

Now we apply Lemma (2.4). For the element $z > 0$,

$$z \wedge [L\backslash(z \wedge L)] \subset F(z \wedge L) = z \wedge L\backslash(z \wedge L)^{o}.$$

By hypothesis there exists a non-zero disjoint element u of z since $z \neq 1$. We claim that $u \in L\backslash(z \wedge L)$. Indeed, if $u \in z \wedge L$, then $u \leq z$, i.e., $z \wedge u = u \neq 0$, which is a contradiction because $z \wedge u = 0$. Therefore, we have

$$0 = z \wedge u \in z \wedge [L\backslash(z \wedge L)] \subset F(z \wedge L)$$

and it follows that $0 \notin (z \wedge L)^{o}$. On the other hand, $0 \in V \subset C_{o,z} = z \wedge L$. Thus, $0 \in (z \wedge L)^{o}$. So we have a contradiction. Hence, cd(L) is not finite.

Immediately we have the following corollary:

(2.7) Corollary. If L is a non-degenerate compact connected topological Boolean lattice, then cd(L) is infinite.

We recall that an element a of a lattice L is neutral if and only if every triple $\{a, x, y\}$ generates a distributive sub-lattice of L and we also recall that an element a center element if and only if it is neutral and complemented in L. We shall write $\mathrm{Cen}(L) = \{x \in L | x$ is a center element} and read "the center of L." Clearly, if L is a distributive lattice with 0 and 1, then the center of L is the set of all $x \in L$ such that for some $y \in L$, $x \wedge y = 0$ and $x \vee y = 1$. Also, if $x \in \mathrm{Cen}(L)$, then there is a unique complemented element of x.

In [10], A. D. Wallace conjectured that the center of a compact connected topological lattice of dimension n contains at most 2^n elements.

Anderson established the validity of the conjecture [3] in the following:

(2.8) Corollary. If L is a compact connected distributive topological lattice and if $\mathrm{cd}(L) = n$, then $\mathrm{Card}(\mathrm{Cen}(L)) \leq 2^n$.

It is now clear that Corollary (2.7) follows from Anderson's result. For, if L is a non-degenerate compact connected Boolean topological lattice of dimension n, we know from (2.8) that $\mathrm{Card}\,(\mathrm{Cen}(L)) \leq 2^n$. But every element of a Boolean lattice is complemented; hence, $L = \mathrm{Cen}(L)$, and therefore L is finite. But L was assumed to be non-degenerate, Hausdorff, and connected, and is consequently infinite, a contradiction.

In 1947 I. Kaplansky [5] showed that:

(2.9) Theorem. A compact semi-simple topological ring is iseomorphic (isomorphic and homeomorphic) with a Cartesian direct sum of finite simple rings.

A subset S of a topological ring A is right bounded if for any neighborhood of the zero 0 of A there exists a neighborhood V such that $VS \subset U$ where $VS = \{vs | v \in V$ and $s \in S\}$.

In the proof of Theorem (2.9), Kaplansky used the following lemma:

(2.10) Lemma. If C is the component of 0 in a right bounded locally compact topological ring A, then $CA = \{0\}$.

This lemma has been proved by using a character group in the sense of topological groups.

It is well known that a Boolean topological lattice becomes a topological ring under the operations of addition and multiplication defined as follows:

$$x \cdot y = x \wedge y \text{ and } x + y = (x \wedge c(y)) \vee (c(x) \wedge y).$$

Moreover, the ring is idempotent. It is easy to show that this ring is always semi-simple and is simple if and only if the Boolean lattice is the Boolean algebra (denoted by 2) consisting of the two elements 0 and 1.

For these reasons, Theorem (2.9) implies the following theorem:

(2.11) Theorem. If L is a compact Boolean topological lattice, then L is iseomorphic to a Cartesian direct sum of the Boolean algebras 2, where 2 has the discrete topology.

Since the Cartesian direct sum (product) of totally disconnected spaces is totally disconnected, and 2 is totally disconnected, Theorem (2.11) implies that any compact Boolean topological lattice is totally disconnected.

Professor A. D. Wallace has suggested the possibility of a proof of this theorem which does not involve the use of duality theorems in the sense of topological groups. In this connection, we prove the following:

(2.12) Corollary. Any compact Boolean topological lattice of finite codimension is totally disconnected.

Proof. Let L be a compact Boolean topological lattice of finite codimension and let C be the component of 0. Now if $x \in C$, we have $0 \in x \wedge C$ and $x \wedge C$ is connected; hence, $x \wedge C \subset C$ and therefore $C \wedge C \subset C$. Dually, $C \vee C \subset C$, and since components are closed, we have that C is a compact sub-lattice of L.

By (2.6), C has a maximal element m. We show $C = m \wedge L$. That $C \subset m \wedge L$ is obvious. If $x \in m \wedge L$, then $x \wedge C$ is connected and $0 \in x \wedge C$. So $x = x \wedge m \in x \wedge C \subset C$ and hence $m \wedge L \subset C$.

It is known that any Boolean lattice is relatively complemented. So a closed interval $C = m \wedge L$ is a compact connected Boolean topological lattice under its relative topology. It is also true that if X is a topological space with $cd(X) = n$ and if C is either an open or a closed subset of X, then $cd(C) \leq n$ [22]. Therefore, C is a compact connected Boolean topological lattice of finite codimension. By (2.7), our only possibility is that C consists of the single element 0. Since L is a topological group, it is homogeneous, and hence L is totally disconnected, which completes the proof.

If we are allowed to use the duality theorem, we can replace the compactness hypothesis in Theorem (2.9) by suitable local properties of L.

We recall that if L is a topological lattice and if for an arbitrary neighborhood U of an element x in L there exists a convex open neighborhood V of x such that $V \subset U$, then L is called locally convex. In other words, the set of convex open neighborhoods of each point of L is a base for the neighborhood system of this point [6].

(2.13) Lemma. If L is a Boolean topological lattice and if L is locally convex, then L is bounded.

Proof. Let U be a neighborhood of 0. Then there exists a locally convex open neighborhood V of 0 such that $V \subset U$. Clearly, by the convexity of V, $V \wedge L \subset V$ because if $v \wedge x \in V \wedge L$, then $0 \leq v \wedge x \leq v$ implies $v \wedge x \in V$. Hence, $V \wedge L \subset U$. Since the operation $\wedge$ is the multiplicative operation in the Boolean ring, L is bounded.

(2.14) Theorem. If L is a locally compact and locally convex Boolean topological lattice, then L is totally disconnected.

Proof. By Lemma (2.13), L is bounded. By Lemma (2.10), if C is the component of 0, then $C \wedge L = \{0\}$. If $x \in C$, then $x = x \wedge x \in C \wedge L = \{0\}$; hence, $x = 0$, i.e., $C = \{0\}$. But L is still a topological group, and hence, L is totally disconnected.

A lattice L is modular if and only if $b \leq a$ implies $b \vee (x \wedge a) = (b \vee x) \wedge a$ for all elements $x \in L$.

We now exhibit a sufficient condition for a compact complemented modular topological lattice to be totally disconnected.

In a lattice, two closed intervals $[a \wedge b, a]$ and $[b, a \vee b]$ are called transposes of each other [2]. Two closed intervals, $[a,b]$ and $[c,d]$, are projective [2] if and only if there exists a finite number of closed intervals $[x_i, y_i]$ such that $[x_{i-1}, y_{i-1}]$ and $[x_i, y_i]$ are transposes, $i = 1, 2, \ldots, n$, where $[a,b] = [x_o, y_o]$ and $[x_n, y_n] = [c,d]$.

(2.15) Lemma. If L is a modular topological lattice, then any pair of projective intervals are topologically equivalent.

Proof. We need to prove only that two transposes $[a \wedge b, a]$ and $[b, a \vee b]$ are topologically equivalent. Define a mapping f from $[a \wedge b, a]$ into $[b, a \vee b]$ by $f(x) = x \vee b$, and define a mapping g from $[b, a \vee b]$ into $[a \wedge b, a]$ by $g(x) = a \wedge y$. Clearly, f and g are both continuous. We show that $g = f^{-1}$, i.e., fg and gf are identities. In fact, for an element y of $[b, a \vee b]$

$$fg(y) = f(a \wedge y) = b \vee (a \wedge y).$$

By the modularity of L, we have $fg(y) = y$ for all $y \in [b, a \vee b]$. Similarly, gf is the identity. Hence, the proof is complete.

(2.16) Lemma. In a topological lattice with 0, the following are equivalent:

(i) $a \wedge L \neq F(a \wedge L)$

(ii) $(a \wedge L)^{o} \neq \square$

(iii) $0 \in (a \wedge L)^{o}$

and dually.

Proof. (i) → (ii) is obvious because $a \wedge L$ is closed.

(ii) → (iii): Anderson has shown [1] that in a topological lattice $(a \wedge L) \vee F(a \wedge L) = F(a \wedge L)$ for all $a \in L$. Suppose that there exists an element x such that $x \in (a \wedge L)^{o}$ and $0 \notin (a \wedge L)^{o}$. It follows that $0 \in F(a \wedge L)$ because $0 \in a \wedge L$. However, $x = x \vee 0 \in (a \wedge L) \vee F(a \wedge L) = F(a \wedge L)$, which is a contradiction. Hence, $0 \in (a \wedge L)^{o}$.

(iii) → (i) is trivial since $0 \notin F(a \wedge L)$; and dually.

For each element x of a topological lattice L with 0 and 1 we define M(x) and L(x) to be, respectively,

$M(x) = x \vee L(=[x,1])$ and $L(x) = x \wedge L(=[0,x])$.

If L is a compact complemented modular topological lattice, then clearly each of the sub-lattices M(x) and L(x) is also a compact complemented modular topological lattice in its relative topology.

If A and B are subsets of M(x) and L(x), respectively, we use $F_M(A)$ and $F_L(B)$ to denote the relative boundaries of A and B, and for $p \in M(x)$ $(p \in L(x))$ we note that $p \wedge M(x) = [x,p]$ $(p \vee L(x) = [p,x])$.

(2.17) Lemma. If L is a compact complemented modular topological lattice and if for every non-zero element a of L $a \wedge L \neq F(a \wedge L)$ and dually, then for every non-unit element x and every element $p(\neq x)$ of $M(x)$, $p \wedge M(x) \neq F_M(p \wedge M(x))$ and dually.

Proof. Since x is not the unit element of L, a complement y of x is not the zero element. Hence, the two closed intervals $[0,y]$ and $[x,1]$ are transposes. So, by Lemma (2.15), $[0,y]$ and $[x,1]$ are topologically equivalent under the mapping $f: [0,y] \to [x,1]$ defined by $f(a) = a \vee x$. Now if $p \in M(x)\backslash x$ and $u \in [0,p \wedge y]$, then $f(u) = u \vee x \in [x,p]$, since $x \leq x \vee u \leq x \vee (p \wedge y) \leq p$; therefore, $f([0,p \wedge y]) \subseteq [x,p]$. Conversely, if $z \in [x,p]$, then $f(y \wedge z) = x \vee (y \wedge z) = z$ by the modularity of L, and therefore, $y \wedge z \in [0,p \wedge y]$. Hence, $f([0,p \wedge y]) = [x,p]$.

Since $p \neq x$ and p is the image of $p \wedge y$ under f, we have $p \wedge y \neq 0$. Now $((p \wedge y) \wedge L)^{\circ} \neq \square$ in L by hypothesis and Lemma (2.16). Therefore, $((p \wedge y) \wedge L)^{\circ} \neq \square$ in $[0,y]$. Thus, $[0,p \wedge y] = (p \wedge y) \wedge L$ implies $(f([0,p \wedge y]))^{\circ} \neq \square$ in $M(x)$ $(=[x,1])$. Applying Lemma (2.16) to $M(x)$, we have that in $M(x)$, $p \wedge M(x) \neq F_M(p \wedge M(x))$ for each $p(\neq x) \in M(x)$. Dually, in $L(x)$, $q \vee L(x) \neq F_M(q \vee L(x))$ for each $q(\neq x) \in L(x)$. Thus the proof is complete.

(2.18) Lemma. If L is a compact complemented modular topological lattice and if for every non-zero element a of L, $a \wedge L \neq F(a \wedge L)$ and dually, then the components of the zero and the unit consist of the single elements $\{0\}$ and $\{1\}$, respectively.

Proof. Let C be the component of 0. We have already shown that C is a compact topological lattice in its relative topology, and $C = p \wedge L$, where p is the maximal element of C. Therefore, C is also a connected complemented topological lattice in its relative topology since any sub-lattice of a complemented modular lattice is also relatively complemented.

Suppose $p \neq 0$. Since there exists an arc from p to 0, we can find an element a such that $0 < a < p$. Applying Lemma (2.4) to C we have that $a \in C$ implies $a \wedge [C \setminus (a \wedge C)] \subset F_C(a \wedge C)$. Let b be a relative complemented element of a in $C = [0,p]$, i.e., $a \wedge b = 0$ and $a \vee b = p$. Thus $b \neq 0$ because $b = 0$ implies $a \vee b = a = p$. Therefore, $b \in C \setminus a \wedge C$, since if $b \in a \wedge C$, we have $b \leq a$, i.e., $b \wedge a = b \neq 0$, which is a contradiction. Consequently,

$$0 = a \wedge b \in a \wedge [C \setminus a \wedge C] \subset F_C(a \wedge C)$$

which implies $0 \notin (a \wedge C)^{o}$. On the other hand, by hypothesis $0 \in [0,a]^{o} = (a \wedge C)^{o}$, which is a contradiction. Hence, p must be the zero. It follows that C consists of the single element 0.

Dually, the component of 1 consists of the single element 1.

Now we prove the following theorem:

(2.19) Theorem. If L is a compact complemented modular topological lattice and if $a \wedge L \neq F(a \wedge L)$ for all non-zero elements a of L and dually, then L is totally disconnected.

Proof. By applying Lemma (2.18) to $M(x)$ $(x \neq 1)$ and $L(y)$ $(y \neq 0)$, the components of x and y in $M(x)$ and $L(y)$ are the singletons $\{x\}$ and $\{y\}$, respectively.

Now consider the component E_x of x in L where $x \neq 1$ and $x \neq 0$. We claim that $E_x \cap (x \vee L) = \{x\}$ and $E_x \cap (x \wedge L) = \{x\}$. In fact, if $y \in E_x \cap (x \vee L)$ and $y \neq x$, then $y < x$. Since E_x is connected, $x \vee E_x$ is connected and contains both x and y. On the other hand, $x \vee E_x \subset x \vee L \subset M(x)$, which is a contradiction. Hence $E_x \cap (x \vee L) = \{x\}$. Dually, $E_x \cap (x \wedge L) = \{x\}$. Consequently, E_x consists of incomparable elements of L. For, if a and b are in E_x and if $a \leq b$, then $E_x = E_a = E_b$. Since x and a are incomparable, a is neither the zero nor the unit element of L. Clearly, $b \in E_b \cap (a \vee L) = E_a \cap (a \vee L) = \{a\}$. Therefore, we have $a = b$.

Now we will show that E_x consists of the singleton $\{x\}$ in L. Suppose $\mathrm{Card}(E_x) > 1$. It follows that there exists an element a of E_x such that a and x are incomparable, and hence, $x \vee a \neq a$, i.e., $x \vee a > a$. Since $x \vee E_x$ is a connected set containing x, $x \vee E_x \subset E_x$. However, $x \vee a \in x \vee E_x \subset E_x$, which is contrary to the fact that E_x consists of incomparable elements. Hence, $E_x = \{x\}$ for all non-zero non-unit elements x of L. For the zero and the unit elements we have already shown in (2.18) that their components consist of the singletons $\{0\}$ and $\{1\}$, respectively. Hence, the proof is complete.

SECTION III

TOPOLOGICAL LATTICES AND INTRINSIC TOPOLOGIES

In this section we study some of the relationships between topological lattices and their intrinsic topologies; for example, the interval topology and the order topology.

By the interval topology of a partially ordered set L, denoted by $\mathcal{I}$, we mean that topology defined by taking the closed intervals $\{[-\infty,a], [a,\infty] | a \in L\}$ as a sub-base for the closed sets [2], where
$[-\infty,a] = \{x|x \in L \text{ and } x \leq a\}$, $[a,\infty] = \{x|x \in L \text{ and } a \leq x\}$. Clearly, if L is a lattice, then $[-\infty,a] = a \wedge L$ and $[a,\infty] = a \vee L$.

The interval topology has been investigated by many authors, mainly Birkhoff [2], Frink [7], Northam [8], Ward [25], Baer [9], and Holland [17].

The following lemma will be needed in the sequel:

(3.1) Lemma. If $(L,\mathcal{T})$ is a compact topological lattice, then L is a complete lattice.

Proof. By Lemma (2.1) all closed intervals $a \wedge L$ and $a \vee L$ are closed in the topology $\mathcal{T}$. It follows from the definition of interval topology that $\mathcal{I} \subset \mathcal{T}$, which means that every subset closed with respect to $\mathcal{I}$ is always a subset closed with respect to $\mathcal{T}$. Since $\mathcal{T}$ is compact, it is not difficult to see that $\mathcal{I}$ is also compact. However, Frink [7] and Ward [25] have shown that $\mathcal{I}$ is compact if and only if L is a complete lattice. Hence, the conclusion follows immediately.

(3.2) Theorem. Let L be Hausdorff in $\mathcal{I}$ and let $(L,\mathcal{T})$ be a topological lattice. $(L,\mathcal{T})$ is compact if and only if $\mathcal{T}=\mathcal{I}$ and L is a complete lattice.

Proof. Suppose that $(L,\mathcal{T})$ is compact. By (3.1), the topologies $\mathcal{T}$ and $\mathcal{I}$ are both compact and Hausdorff. Now consider the inclusion mapping $i : (L\,\mathcal{T}) \rightarrow (L,\mathcal{I})$. Clearly i is continuous and onto since $\mathcal{I} \subset \mathcal{T}$. Therefore, it follows that i is a homeomorphism. Hence $\mathcal{T} = \mathcal{I}$. Again, by (3.1), L is a complete lattice.

Conversely, suppose that $\mathcal{T} = \mathcal{I}$ and L is a complete lattice. It follows from the results of Frink and Ward that $\mathcal{I}$ is compact. So $\mathcal{T}$ is compact, i.e., $(L,\mathcal{T})$ is a compact topological lattice.

Let L be a lattice with 0. L is atomic if and only if for every non-zero element x of L there exists an element p of L such that $x \geq p$ and p covers 0, where by p covers 0 we mean that there is no element q such that $p > q > 0$. Such an element p is called an atom of L. A subset M of a lattice L is an ideal of L if and only if $M \neq \square$, $M \vee M \subset M$ and $M \wedge L \subset M$. A dual-ideal M of L means that $M \neq \square$, $M \wedge M \subset M$ and $M \vee L \subset M$.

(3.3) Corollary. If L is a topological lattice with 0 and if every maximal dual ideal is compact, then L is atomic.

Proof. Let x be a non-zero element of L. By the Hausdorff Maximality Principal, there is a maximal dual ideal M containing x. Since M is a compact topological lattice in its relative topology, M has its zero element p. If $p = 0$, $M = L$ because $a = 0 \vee a \in M \vee L \subset M$ for all $a \in L$. Hence, $p \neq 0$. Furthermore, there is no element q of L such that $0 < q < p$, for otherwise, we have a dual ideal $q \vee L$ such that $q \vee L \supset M$ and $M \neq q \vee L \neq L$, which is a contradiction. Hence, L is atomic.

Now we define the order topology of a complete lattice. A directed set is a pair $(D,\geq)$ such that D is a non-void set with a binary relation $\geq$ which is transitive, reflexive, and if α and β are members of D, then there exists a γ in D such that $\gamma \geq \alpha$ and $\gamma \geq \beta$.

$\{x_\alpha | \alpha \in D\}$ is a net in a set A if and only if D is a directed set and $x_\alpha \in A$ for all $\alpha \in D$.

Let $\{x_\alpha | \alpha \in D\}$ be a net in a complete lattice L. We say that x_α order-converges to an element a of L if

$$\mathrm{Inf}_\alpha\{\mathrm{Sup}_{\beta\geq\alpha}\, x_\beta\} = \mathrm{Sup}_\alpha\{\mathrm{Inf}_{\beta\geq\alpha}\, x_\beta\} = a.$$

Setting $u_\alpha = \mathrm{Sup}_{\beta\geq\alpha}\, x_\beta$ and $v_\alpha = \mathrm{Inf}_{\beta\geq\alpha}\, x_\beta$, we know that x_α order-converges to a if and only if there exist a monotone decreasing net $\{u_\alpha | \alpha \in D\}$ with $\mathrm{Inf}_\alpha\, u_\alpha = a$ and a monotone increasing net $\{v_\alpha | \alpha \in D\}$ with $\mathrm{Sup}_\alpha\, v_\alpha = a$ such that $v_\alpha \leq x_\alpha \leq u_\alpha$ for all $\alpha \in D$.

We define a subset M of a complete lattice L to be closed in the order topology of L, denoted by $\mathcal{O}$, if and only if $\{x_\alpha | \alpha \in D\}$ is a net in M and $\{x_\alpha | \alpha \in D\}$ order-converges to a implies that $a \in M$.

It is well known that any order topology has the T_1-separation property, and $\mathcal{I} \subset \mathcal{O}$, where $\mathcal{I}$ is the interval topology of L. It is also well known [6] that any compact partially ordered topological space is locally convex, which means that the set of convex neighborhoods of each point of the space is a base for the neighborhood system of this point. And it is also known that any topological lattice is a continuous partially ordered topological space [11, 12, 13].

Now we shall prove the following lemma:

(3.4) Lemma. If $(L, \mathcal{T})$ is a compact topological lattice, and if $\{x_\alpha | \alpha \in D\}$ is a monotone decreasing (or increasing) net in L with $\mathrm{Inf}_{\alpha \in D}\, x_\alpha = a$ (or $\mathrm{Sup}_{\alpha \in D}\, x_\alpha = a$), then the net $\{x_\alpha | \alpha \in D\}$ converges to a in $\mathcal{T}$.

Proof. Let $\{x_\alpha | \alpha \in D\}$ be a monotone decreasing net with $\mathrm{Inf}_{\alpha \in D}\, x_\alpha = a$. Suppose that there exists an open neighborhood U of a such that $x_\alpha \notin U$ for all $\alpha \in D$. Since $a \vee L$ is closed, $(a \vee L)\backslash U$ is closed. $\mathrm{Inf}_{\alpha \in D}\, x_\alpha = a$ implies $x_\alpha \geq a$ for all $\alpha \in D$. So $x_\alpha \in (a \vee L)\backslash U$ for all $\alpha \in D$. Consider the family

$$\{(x_\alpha \wedge L) \cap [(a \wedge L)\backslash U] | \alpha \in D\}.$$

Since $x_\alpha \in (x_\alpha \wedge L) \cap [(a \wedge L)\backslash U]$, each member of the family is not empty. We now show that the family has the finite intersection property. In fact, for an arbitrary finite subset $F = \{p, q, \ldots, t\}$ in D, there exists $\beta \in D$ such that $\beta \geq p, \ldots, \beta \geq t$ in D. By hypothesis, the net $\{x_\alpha | \alpha \in D\}$ is monotone decreasing, and hence, $x_\beta \leq x_p, \ldots, x_\beta \leq x_t$. Therefore, it follows that $x_\beta \wedge L \subset x_p \wedge L, \ldots, x_\beta \wedge L \subset x_t \wedge L$, and so $\square \neq (x_\beta \wedge L) \cap [(a \vee L)\backslash U] \subset \cap_{r \in F} (x_r \wedge L) \cap [(a \vee L)\backslash U]$. Since L is compact, there exists an element b such that

$$b \in \cap_{\alpha \in D} \{(x_\alpha \wedge L) \cap [(a \vee L)\backslash U]\}.$$

Thus, $b \leq x_\alpha$ for all $\alpha \in D$, hence, $b \leq \mathrm{Inf}_\alpha\, x_\alpha = a$. However, $b \in (a \vee L)\backslash U$ implies $b \geq a$. Hence, $a = b$. This is a contradiction. Thus, $x_\alpha \in U$ for some $\alpha \in D$.

Using the above argument, we may assume that the neighborhood U of a is convex, since L is compact and hence, locally convex. It follows that $x_\beta \in U$ for all $\beta \geq \alpha$ because $x_\alpha \in U$ and $a \leq x_\beta \leq x_\alpha$ for all $\beta \geq \alpha$. Therefore, $\{x_\alpha | \alpha \in D\}$ converges to a. Dually, we can show that any monotone increasing net $\{x_\alpha | \alpha \in D\}$ with $\mathrm{Sup}_\alpha\, x_\alpha = a$ converges to a. Thus, the proof is complete.

(3.5) Lemma. If $(L, \mathcal{T})$ is a compact topological lattice, then $\mathcal{T} \subset \mathcal{O}$.

Proof. Suppose that a net $\{x_\alpha | \alpha \in D\}$ in L order-converges to an element a of L, i.e.,

$$\mathrm{Sup}_\alpha \{\mathrm{Inf}_{\beta \geq \alpha}\, x_\beta\} = \mathrm{Inf}_\alpha \{\mathrm{Sup}_{\beta \geq \alpha}\, x_\beta\} = a.$$

Setting $v_\alpha = \mathrm{Inf}_{\beta \geq \alpha}\, x_\beta$ and $u_\alpha = \mathrm{Sup}_{\beta \geq \alpha}\, x_\beta$, we have that $v_\alpha \leq x_\alpha \leq u_\alpha$ for all $\alpha \in D$, and $\{v_\alpha | \alpha \in D\}$ and $\{u_\alpha | \alpha \in D\}$ are monotone increasing and decreasing nets with $\mathrm{Sup}_\alpha\, v_\alpha = a$

and $\mathrm{Inf}_\alpha\, u_\alpha = a$, respectively.

It follows by (3.4) that both nets $\{u_\alpha | \alpha \in D\}$ and $\{u_\alpha | \alpha \in D\}$ converge to a in $\mathcal{T}$. Therefore, for an arbitrary convex neighborhood U of a there exist α and β in D such that v_α and u_β are elements of U. Since v_δ and u_δ are elements of U for all $\delta \geq \alpha$ and $\delta \geq \beta$, if we take an element $\gamma \in D$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$ in D, then v_γ and u_γ are elements of U. Hence, we have $x_\delta \in U$ for all $\delta \geq \gamma$ since $u_\gamma \leq x_\gamma \leq u_\gamma$. So it follows that $\{x_\alpha | \alpha \in D\}$ converges to a in $\mathcal{T}$. Hence, we have $\mathcal{T} \subset \mathcal{O}$. Indeed, let S be a closed subset of L with respect to $\mathcal{T}$, and let $\{x_\alpha | \alpha \in D\}$ be a net in S such that $\{x_\alpha | \alpha \in D\}$ order-converges to a. By the above, $\{x_\alpha | \alpha \in D\}$ converges to a in $\mathcal{T}$, and hence, $a \in S$. Thus, S is a closed subset of L with respect to $\mathcal{O}$.

Using Lemma (3.5), we have the following theorem:

(3.6) Theorem. If L is compact in its order topology $\mathcal{O}$ and if $(L, \mathcal{T})$ is a compact topological lattice, then $\mathcal{O} = \mathcal{T}$.

Proof. By Lemma (3.5), $\mathcal{T} \subset \mathcal{O}$. If we consider again the inclusion mapping $i : (L, \mathcal{O}) \rightarrow (L, \mathcal{T})$, then i is clearly continuous and onto. Since $\mathcal{T}$ is Hausdorff and $\mathcal{O}$ is compact, i is a homeomorphism; hence, $\mathcal{O} = \mathcal{T}$.

By the pseudo-complement c of an element a relative to an element b in a lattice L is meant an element, denoted by $c = a * b$, such that $a \wedge x \leq b$ if and only if $x \leq c$. If for all a and b in L there is the pseudo-complement $a * b$ in L, then L is called relatively pseudo-complemented [2].

In [2], Theorem 5, pp. 147, says that a complete lattice L is relatively pseudo-complemented if and only if L satisfies the infinite distributive law:

$$x \wedge (\vee B) = \vee (x \wedge B) \text{ for all subsets } B \text{ of } L,$$

where $\vee B = \mathrm{Sup}_{b \in B}\, b$ and $\wedge A = \mathrm{Inf}_{a \in A}\, a$.

Now we show that the following corollary holds:

(3.7) Corollary. Let L be a compact topological lattice. L is relatively pseudo-complemented if and only if L is distributive.

Proof. L being compact, we have that L is a complete lattice. So to show this corollary we need only to show that distributivity of L implies infinite distributivity. Let B be a non-void subset of L and let I be the set of all finite subsets G of B. We show that for an element x of L

$$x \wedge (\vee B) = \vee (x \wedge B).$$

Clearly, I is a directed set under the set inclusion relation. Setting $z_G = \vee G$, $\{z_G | G \in I\}$ is a monotone increasing net with $\vee z_G = \vee B$. According to (3.4) we have that $\{z_G | G \in I\}$ converges to $\vee z_G$. Since the meet-lattice operation of L is continuous, $\{x \wedge z_G | G \in I\}$ converges to $x \wedge (\vee z_G) = x \wedge (\vee B)$. However, $x \wedge z_G = x \wedge (\vee G) = \vee (x \wedge G)$ because G is finite and L is distributive. Setting $u_G = \vee (x \wedge G)$, $\{u_G | G \in I\}$ is seen to be a monotone increasing net with $\vee u_G = \vee (x \wedge B)$. Again, by (3.4) the net $\{u_G | G \in I\}$ converges to $\vee u_G = \vee (x \wedge B)$.

On the other hand, $u_G = \vee (x \wedge G) = x \wedge z_G$ for all

$G \in I$. Therefore, $\{x \wedge z_G | G \in I\}$ and $\{u_G | G \in I\}$ are exactly the same nets and hence, their limit points $x \wedge (\vee B)$ and $\vee (x \wedge B)$ must be the same since L is Hausdorff. So we have $x \wedge (\vee B) = \vee (x \wedge B)$ for all $x \in L$, which completes the proof.

E. Floyd [15] has shown that a complete Boolean algebra is not necessarily Hausdorff in the order topology. Now we shall find a sufficient condition for a complete Boolean algebra to be a Hausdorff space in its order topology.

(3.8) Theorem. If L is a complete Boolean algebra and if the join-lattice operation (or the meet-lattice operation) of L is continuous in its order topology, then L is a Hausdorff space in its order topology.

Proof. First of all we show that the unary operation of complementation of L is always continuous in its order topology. For an element x of L, let x' be the complemented element of x. For an arbitrary open neighborhood U of x' we claim that the subset $U' = \{y' | y \in U\}$ is also open and $x \in U'$. For, if $\{x_\alpha | \alpha \in D\}$ is a net in $L \setminus U'$ and if $\{x_\alpha | \alpha \in D\}$ order-converges to an element a of L, then

$$\mathrm{Sup}_\alpha\{\mathrm{Inf}_{\beta \geq \alpha}\, x_\beta\} = \mathrm{Inf}_\alpha\{\mathrm{Sup}_{\beta \geq \alpha}\, x_\beta\} = a.$$

On the other hand, it is well known that for a subset A of a Boolean algebra

$$(\mathrm{Inf}\ A)' = \mathrm{Sup}\ A' \quad \text{and} \quad (\mathrm{Sup}\ A)' = \mathrm{Inf}\ A'$$

if Sup A and Inf A exist. Thus, it immediately follows that

$$\mathrm{Inf}_{\alpha}\{\mathrm{Sup}_{\beta\geq\alpha}\ x'_{\beta}\} = \mathrm{Sup}_{\alpha}\{\mathrm{Inf}_{\beta\geq\alpha}\ x'_{\beta}\} = a'.$$

Since $x_{\alpha} \in L\backslash U'$ for all $\alpha \in D$, $x'_{\alpha} \in L\backslash U$ for all $\alpha \in D$. So $\{x'_{\alpha} | \alpha \in D\}$ is a net in the closed subset $L\backslash U$ and order-converges to a'. Thus, $a' \in L\backslash U$. It follows that $a \in L\backslash U'$, and hence, $L\backslash U'$ is closed, i.e., U' is open and $x \in U'$. Hence, the complementation operation is continuous.

Now suppose that the join-operation is continuous in its order topology. In other words, a mapping g from $L \times L$ into L defined by $g(a,b) = a \vee b$ is continuous for all pairs of (a,b) in L. Let f be the mapping from L into L defined by $f(a) = a'$ for all elements a in L and let h be a mapping from $L \times L$ into L defined by $h(a,b) = a \wedge b$ for all pairs of a and b in L. In a Boolean algebra f^2 is the identity and $f(f(a) \vee f(b)) = a \wedge b$.

Now consider the following diagram:

$$\begin{tikzcd}
L \times L \arrow[r] \arrow[d, "f \times f"] & L \\
L \times L \arrow[r] & L \arrow[u, "f"']
\end{tikzcd}$$

where $(f\times f)(a,b) = (f(a), f(b))$ for all pairs of a and b in L. It is easy to see that this diagram is commutative. Indeed,

$$fg(f\times f)(a,b)=fg(f(a),f(b))=f(f(a)\vee f(b))=a\wedge b=h(a,b).$$

Therefore, h is continuous. It follows that $(L, \mathcal{O})$ is a topological group under the symmetric difference operation +

which is defined by $a + b = (a' \wedge b) \vee (a \wedge b')$ for all elements a and b in L. Since the order topology $\mathcal{O}$ is always a T_1-space, the topological space $(L, \mathcal{O})$ is regular. It follows that $(L, \mathcal{O})$ is Hausdorff. Thus, the proof is complete.

We shall be concerned below with lattice-ordered groups, denoted by l-group.

An l-group $L(+, \wedge, \vee)$ [2], denoted by L, is a quadruple such that

(i) L is a group under the operation +,

(ii) L is a lattice under the operations $\wedge$ and $\vee$,

(iii) for elements a, b, x and y in L $a+(x \vee y) = (a + x) \vee (a + y)$ and $(x \vee y) + b = (x + b) \vee (y + b)$, and dually.

An element a of an l-group L is called positive if $a \geq 0$.

An l-group has the chain condition if and only if every non-void set of positive elements in L includes a minimal element [2]. In such an l-group, any element which covers 0, the identity of L, will be called a prime.

G. Birkhoff has shown the following theorem [16]:

(3.9) Theorem. Let L be an l-group which satisfies the chain condition. Then L is commutative and each non-zero element of L can be expressed uniquely as a sum of integral multiples of distinct primes, i.e., $a = n_1 p_1 + \ldots + n_k p_k$ for all $a \in L$, where the n's are integers and the p's are prime elements of L. Such a sum is positive if and only if all coefficients n_i are positive.

R. M. Baer [9] has characterized a lattice to be Hausdorff in its interval topology by the following:

(3.10) Theorem. A necessary and sufficient condition that the interval topology of a lattice L be Hausdorff is that for every pair of elements a and b in L with $a < b$, there exist finite non-void subsets A and B (depending on a and b) in L such that both of the following conditions are satisfied:

(i) $A = \{x \mid a < x \leq b$ or x is either incomparable with a or b$\}$,

$B = \{y \mid a \leq y < b$ or y is either incomparable with a or b$\}$,

(ii) $(L \vee A) \cup (L \wedge B) = L$.

Now we will prove the following theorem:

(3.11) Theorem. If L is an l-group which satisfies the chain condition, then L is Hausdorff in its interval topology if and only if L is a chain, i.e., L is a cyclic group.

Proof. The sufficiency is trivial. We now show the necessity. Suppose that L is Hausdorff in the interval topology and L contains more than two primes, say p and q. According to Theorem (3.10), there exist finite non-void subsets A and B of L such that (i) and (ii) of Theorem (3.10) hold for some pair of elements a and b with $a < b$.

By Theorem (3.9), each element of A and B can be expressed uniquely as a sum of integral multiples of a finite number of distinct primes. Let n be the greatest integer of all these integral coefficients and zero, and let m be the least of all these integral coefficients and zero. It is clear that $n + 1 > 0$ and $m - 1 < 0$. Now consider the element $(n + 1)p + (m - 1)q$ of L and we assert that this element does not belong to $(L \vee A) \cup (L \wedge B)$. For, suppose $(n + 1)p + (m - 1)q \in (L \vee A) \cup (L \wedge B)$, i.e., $(n + 1)p + (m - 1)q \in L \vee A$, or $(n + 1)p + (m - 1)q \in L \wedge B$, say, $(n + 1)p + (m - 1)q \in L \vee A$; then there exists an element $a \in A$ such that $a \leq (n + 1)p + (m - 1)q$. It follows that $0 \leq (n + 1)p + (m - 1)q - a$. a can be expressed by $a = n_1p_1 +\ldots+ n_rp_r$, where n_i is an integer with $m \leq n_i \leq n$, and p_i is a prime, $i = 1, 2, \ldots, r$. Thus, integral multiples of p and q in the element $(n + 1)p + (m - 1)q - a$ are still positive and negative, respectively, which is contrary to the fact that $(n + 1)p + (m - 1)q - a \geq 0$. Similarly, the other case also leads to a contradiction. Hence, (ii) of Theorem (3.10) is not satisfied. Therefore, we can conclude that L contains at most one prime element. It follows that L is either a cyclic group generated by the prime element or L consists of single element zero, which completes the proof.

(3.12) Corollary. Let L be an l-group which satisfies the chain condition. L is a topological lattice in its interval topology if and only if L is a chain.

Remark. Recently C. Holland [17] has given an example of an l-group which satisfies neither the chain condition nor is a chain, but forms a topological lattice in its interval topology.

We recall that a subset S of a partially ordered set L is up-directed (down-directed) if and only if for all elements x and y in S there exists $z \in S$ such that $z \geq x$, $z \geq y$ ($z \leq x$, $z \leq y$).

Now we define the Birkhoff-Frink-McShane 0-convergence in a partially ordered set.

A net $\{x_\alpha | \alpha \in D\}$ in a partially ordered set L 0-converges to $y \in L$ if and only if there exist non-void subsets M and N of L such that

(i) M is up-directed and N is down-directed,

(ii) $y = \text{Sup } M = \text{Inf } N$ and

(iii) for each element $m \in M$ and each element $n \in N$ there exists $\beta \in D$ such that $m \leq x_\alpha \leq n$ for all $\alpha \geq \beta$ in D.

We define a subset A of a partially ordered set L to be closed in the 0-topology of L if and only if $\{x_\alpha | \alpha \in D\}$ is a net in A and $\{x_\alpha | \alpha \in D\}$ 0-converges to an element a of L implies that $a \in A$.

(3.13) Lemma. If L is an l-group and if the zero element in L is an isolated point in the 0-topology, then L is a discrete space in its 0-topology.

Proof. By hypothesis $\{0\}$ is an open set in L. If y is an arbitrary element of L and if $L \setminus \{y\}$ is not a closed set of L, then there exists a net $\{x_\alpha | \alpha \in D\}$ in $L \setminus \{y\}$ such that $\{x_\alpha | \alpha \in D\}$ 0-converges to y. Therefore, it follows that there exist two subsets M and N of L satisfying (i), (ii), and (iii) of the definition of 0-convergence. It is easy to see that $M-y = \{m-y | m \in M\}$ and $N-y = \{n-y | n \in N\}$ are up-directed and down-directed, respectively, and $0 = \mathrm{Sup}(M-y) = \mathrm{Inf}(N-y)$ by (ii). From (iii), for each element $m-y \in M-y$ and each element $n-y \in N-y$, there exists an element β of D such that $m-y \leq x_\alpha - y \leq n-y$ for all $\alpha \in D$ with $\alpha \leq \beta$. Since $x_\alpha \neq y$ for all $\alpha \in D$, $\{x_\alpha - y | \alpha \in D\}$ is a net in $L \setminus \{0\}$ and 0-converges to 0 which is a contradiction. Hence, $\{y\}$ is an open set in L which completes the proof.

We recall that an l-group is called complete if and only if every non-void bounded set has a greatest lower bound and a least upper bound.

(3.14) Theorem. Let L be a complete l-group. L is discrete in the 0-topology if and only if L has the chain condition.

Proof. Suppose that L is discrete in its 0-topology. Let A be a non-void subset of positive elements in L. By the Hausdorff Maximality Principle, there exists a maximal chain C in A. Since every element of C is positive, C is bounded below. Therefore, there is an $m = \mathrm{Inf}\ C$ in L. If $m \notin C$, then C must be an infinite chain. Consider $\{c | c \in C\}$ where C is a directed set under the original order relation of L. Then $\{c | c \in C\}$ is a net in $L \setminus \{m\}$ since $m \notin C$. If we take $M = \{m\}$ and $N = C$, then it is easy to see that (i), (ii), and (iii) in the definition of 0-convergence are satisfied. Therefore, the net $\{c | c \in C\}$ 0-converges to m, which is impossible in a discrete space. Hence $m \in C$, and hence, m is a minimal element of A, which is what we required.

To prove the converse, suppose that L satisfies the chain condition. According to Lemma (3.13) we need only to show that the zero element 0 is an isolated point in L. If $L \setminus \{0\}$ is not closed, then there exists a net $\{x_\alpha | \alpha \in D\}$ in $L \setminus \{0\}$ such that $\{x_\alpha | \alpha \in D\}$ 0-converges to 0. Thus L has two subsets M and N satisfying (i), (ii), and (iii) in the definition of 0-convergence. From (iii), we have either $0 \notin N$ or $0 \notin M$. Let us first assume $0 \notin N$. Since $0 = \text{Inf } N$, every element of N is positive, i.e., N is a non-empty set of positive elements. By the chain condition, N has a minimal element n. By (i), n must be the least element of N. Clearly, $n \neq 0$. This is a contradiction to (ii). Now assume $0 \notin M$. It is well known that in any l-group the set of all positive elements and of all negative elements are anti-isomorphic. Hence, the chain condition implies that every non-void set of negative elements has a maximal element. Thus, by the dual argument, M has a greatest element, which is also impossible. Therefore, L is discrete.

SECTION IV

COMPACT AND CONNECTED DISTRIBUTIVE TOPOLOGICAL LATTICES

In this section another method of proving Wallace's conjecture, stated in Section II, will be given and the representation theorems of some distributive topological lattices as n-cells will also be considered. In addition, several structure theorems of compact topological lattices are proved.

We use the term iseomorphism to mean a mapping which is simultaneously an isomorphism and a homeomorphism and use the symbol $c(x)$ to denote the complement element of x in a lattice with 0 and 1 or in a Boolean lattice.

The following theorem is due to Anderson [3]:

(4.1) Theorem. If L is a distributive (chain-wise connected) topological lattice, then $Br(L) \leq n$ if and only if L does not contain a sub-lattice topologically isomorphic with a Cartesian product of $n + 1$ non-degenerate (closed and connected) chains.

The following lemma follows from the proof of Theorem 10 in [5]:

(4.2) Lemma. If L is a totally disconnected compact Boolean topological lattice, then L is iseomorphic with the set of all subsets of a set X denoted by 2^X.

(4.3) Theorem. If L is a compact, connected, distributive topological lattice and if $cd(L) \leq n$, then $\mathrm{Card}(\mathrm{cen}(L)) \leq 2^n$.

Proof. Let S be the set of all center elements in L. Since L is distributive, $x \in S$ if and only if x has a complement element $c(x)$ in L; $x \wedge c(x) = 0$ and $x \wedge c(x) = 1$. Thus, clearly, S forms a Boolean sub-lattice with 0 and 1 in L. We show that S is a closed subset of L. For, we may define mappings:

$$\varphi : L \times L \to L \times L \text{ by } \varphi(x,y) = (x \wedge y,\ 0) \text{ and}$$

$$\psi : L \times L \to L \times L \text{ by } \psi(x,y) = (1,\ x \vee y).$$

Obviously, φ and ψ are both continuous. Therefore, it follows that $\varphi^{-1}(\Delta) \cap \psi^{-1}(\Delta)$ is closed in $L \times L$, where $\Delta = \{(x,x) | x \in L\}$, which is called the diagonal of L. Clearly,

$\varphi^{-1}(\Delta) \cap \psi^{-1}(\Delta) = \{(x,y) \in L \times L | x \wedge y = 0 \text{ and } x \vee y = 1\}$.

Let π be the projection mapping of $L \times L$ into the first L. It is easy to see that $S = \pi(\varphi^{-1}(\Delta) \cap \psi^{-1}(\Delta))$. L being compact, $\pi(\varphi^{-1}(\Delta) \cap \psi^{-1}(\Delta))$ is closed in L so that S is closed. Hence, S is a compact Boolean topological lattice of finite codimension. By Corollary (2.12), S is a totally disconnected, compact, Boolean topological lattice.

It follows from Lemma (4.2) that S is iseomorphic with 2^X for some set X. Suppose that $\mathrm{Card}(X) > n$. We can take, then, $(n+1)$ elements $a_1, a_2, \ldots, a_{n+1}$ in S such that the iseomorphism from S to 2^X maps a_i to an atom of 2^X.

Thus, each a_i is also an atom in S. Since L is compact and connected, we can find a compact connected chain C_i from 0 to a_i in L, i = 1, 2, ..., n+1. It follows from $a_i \wedge a_j = 0$ $(i \neq j)$ that if $x_i \in C_i$ and $x_j \in C_j$ $(i \neq j)$, then $x_i \wedge x_j = 0$.

Now consider the following mappings:

$$f : C_1 \times \dots \times C_{n+1} \to L \text{ defined by } f(x_1, \dots, x_{n+1}) = x_1 \vee \dots \vee x_{n+1}$$

$$g : f(C_1 \times \dots \times C_{n+1}) \to C_1 \times \dots \times C_{n+1} \text{ defined by } g(x) = (x \wedge a_1, \dots, x \wedge a_i, \dots, x \wedge a_{n+1}).$$

Clearly, both f and g are well defined and continuous. Furthermore, $g = f^{-1}$, i.e., gf and fg are the identities. Thus, L contains a sub-lattice topologically isomorphic with a Cartesian product $C_1 \times \dots \times C_{n+1}$ of non-degenerate closed and connected chains C_i. Therefore, $Br(L) \nleq n$ by Theorem (4.1) which means $n < Br(L) \leq cd(L)$ by Corollary (2.5). This is a contradiction. Hence, $Card(X) \leq n$. It follows that $Card(S) = 2^m$ for some $m \leq n$, i.e., $Card(Cen(L)) = Card(S) \leq 2^n$. This completes the proof.

We now are concerned with a representation of a distributive topological lattice by an n-cell.

The set of n elements $x_1, x_2, \dots, x_n$ in a lattice L is independent over an element a of L if and only if $x_i < a$ and $(x_1 \vee \dots \vee x_k) \wedge x_{k+1} = a$, k = 1, 2, ..., n-1 [2]. It is easy to see that L is distributive then $x_1, \dots, x_n$ are independent over a if and only if $x_i \wedge x_j = a$ $(i \neq j)$.

We will prove the following useful lemma:

(4.4) Lemma. If L is a distributive topological lattice with 0 and 1 and if L contains n independent elements $x_1, \ldots, x_n$ over 0 whose union is 1, then L is iseomorphic with the topological lattice of the Cartesian product $(x_1 \wedge L) \times (x_2 \wedge L) \times \ldots \times (x_n \wedge L)$.

Proof. If we define a mapping $h : L \to (x_1 \wedge L) \times \ldots \times (x_n \wedge L)$ by $h(a) = (a \wedge x_1, \ldots, a \wedge x_n)$, then h is obviously well defined and continuous since each component is continuous. Furthermore, h is a lattice-homomorphism, since for all elements a and b in L,

$$h(a \wedge b) = ((a \wedge b) \wedge x_1, \ldots, (a \wedge b) \wedge x_n)$$
$$= h(a) \wedge h(b) \text{ and } h(a \vee b) = ((a \vee b) \wedge x_1, \ldots,$$
$$(a \vee b) \wedge x_n) = ((a \wedge x_1) \vee (b \wedge x_1), \ldots,$$
$$(a \wedge x_n) \vee (b \wedge x_n)) = h(a) \vee h(b).$$

Now we define a mapping $g : (x_1 \wedge L) \times \ldots \times (x_n \wedge L) \to L$ by $g(a_1, \ldots, a_n) = a_1 \vee a_2 \vee \ldots \vee a_n$. Then, clearly, g is continuous. Furthermore, $g = h^{-1}$. Indeed, by distributivity

$$\begin{aligned} hg(a_1, a_2, \ldots, a_n) &= h(a_1 \vee a_2 \vee \ldots \vee a_n) \\ &= ((a_1 \vee \ldots \vee a_n) \wedge x_1, \\ &\quad \ldots, (a_1 \vee \ldots \vee a_n) \wedge x_n) \\ &= (a_1 \wedge x_1, \ldots, a_n \wedge x_n) \\ &= (a_1, a_2, \ldots, a_n), \end{aligned}$$

$$\begin{aligned} gh(a) &= g(a \wedge x_1, \ldots, a \wedge x_n) = (a \wedge x_1) \vee \ldots \vee (a \wedge x_n) \\ &= a \wedge (x_1 \vee \ldots \vee x_n) = a \wedge (x_1 \vee \ldots \vee x_n) \\ &= a \wedge 1 = a \quad \text{by hypothesis.} \end{aligned}$$

Hence, L is iseomorphic with $(x_1 \wedge L) \times (x_2 \wedge L) \times \ldots \times (x_n \wedge L)$.

(4.5) Theorem. Suppose L is a topological lattice with 0 and 1. Then L is iseomorphic with an n-cell if and only if L satisfies the following two conditions:

(i) L is distributive and contains n independent elements $x_1, x_2, \ldots, x_n$ over 0 whose union is 1.

(ii) Each $x_i \wedge L$ is a separable connected locally convex chain.

Proof. Suppose that there exists an iseomorphism f from L to an n-cell I^n where I is the real unit interval $[0,1]$. Then obviously, L is distributive, since so is I^n. Consider the fundamental base of I^n : $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, ..., $(0, 0, \ldots, 1)$. These elements are clearly independent elements over the zero $(0, \ldots, 0)$ in I^n.

Let $x_1, \ldots, x_n$ be elements of L for which $f(x_1) = (1, 0, \ldots, 0)$, ..., $f(x_n) = (0, 0, \ldots, 0, 1)$. Since f is a lattice isomorphism, these elements $x_1, \ldots, x_n$ are obviously independent elements over 0 in L, and moreover, for each i
$f(x_i \wedge L) = \{(0, \ldots, 0, a_i, 0, \ldots, 0) \in I^n | a_i \in I\}$, and hence, $f(x_i \wedge L)$ is iseomorphic with I for each i. It follows that each $x_i \wedge L$ is a separable connected locally convex chain.

Conversely, suppose that L satisfies the conditions (i) and (ii) in the theorem. It follows from Lemma (4.4) that L is iseomorphic with the Cartesian product $(x_1 \wedge L) \times \ldots \times (x_n \wedge L)$. By (ii) each $x_i \wedge L$ is a separable connected locally convex chain. Now we recall Anderson's result, [18] and [19], that any separable connected locally convex chain is order homeomorphic to the closed interval I. Consequently, it follows that L is iseomorphic with $I \times \ldots \times I = I^n$, which completes the proof.

(4.6) Corollary. If L is a topological lattice with 0 and 1 with $cd(L) = n$, then L is iseomorphic with an n-cell if and only if L satisfies (i) in Theorem (4.5) and (ii) each $x_i \wedge L$ is separable, connected and locally compact.

Before we prove the corollary we give the following lemma:

(4.7) Lemma. If L is a locally compact topological lattice with $cd(L) = n$ and if $C_1, C_2, \ldots, C_m$ are a locally compact connected chains, then $cd(L \times C_1 \times \ldots \times C_m) = n + m$.

This lemma is an immediate consequence of H. Cohen's results [20] that

(i) if X is a locally compact Hausdorff space and if $ind(X) = 1$, then $cd(X) = 1$ where "ind" means the inductive dimension of Urysohn and Menger [21], and

(ii) If X and Y are locally compact Hausdorff spaces and if $cd(X) = n$ and $ind(Y) = 1$, then $cd(X \times Y) = n+1$. Therefore, Lemma (4.7) follows immediately from (ii) since $ind(C_i) = 1$ and the C_i, $i = 1, 2, \ldots, m$, are a locally compact chains.

Proof of Corollary (4.6). The necessity is obvious.

We now prove the sufficiency. By Lemma (4.4), L is iseomorphic with $(x_1 \wedge L) \times \ldots \times (x_n \wedge L)$. Since $x_i \wedge L$ is non-degenerate, connected, and locally compact, we have $cd(x_i \wedge L) \geq 1$ for each i, and since $cd(L) = cd((x_1 \wedge L) \times \ldots \times (x_n \wedge L)) = n$, we can verify that each $cd(x_i \wedge L) = 1$, $i = 1, 2, \ldots, n$. Indeed, suppose $cd(x_i \wedge L) > 1$ for some i. For each $j(\neq i)$, we take a compact connected chain C_j such that $C_j \subset x_j \wedge L$ and $cd(C_j) = 1$. Clearly, by Lemma (4.7), we have that $cd(L) \geq cd(C_1 \times \ldots \times C_{i-1} \times (x_i \wedge L) \times C_{j+1} \times \ldots \times X_n) > n$, which is a contradiction. Hence, $cd(x_i \wedge L) = 1$ for each i, and hence, each $x_i \wedge L$ is a chain [1].

On the other hand, $x_i \wedge L$ is locally convex since it is connected and locally compact [18]. Therefore, by Theorem (4.5), L is iseomorphic with an n-cell. Thus, the proof is complete.

We now consider the cardinality of the center in a distributive lattice with 0 and 1. The following lemma is useful in the sequel.

(4.8) Lemma. Let L be a distributive topological lattice with 0 and 1. If $x_1, \ldots, x_n$ $(n > 1)$ are all atoms of the center in L and if x and y are two incomparable elements in $x_i \wedge L$ for some i, then

$$\mathrm{Card}(\mathrm{Cen}(L)) < \mathrm{Card}(\mathrm{Cen}([x \wedge y, x \vee y \vee c(x_i)])),$$

where $c(x_i)$ is the complemented element of x_i in the center of L.

Proof. Without losing any generality, we can assume that x and y are two incomparable elements in $x_1 \wedge L$. Clearly, $\mathrm{Card}(\mathrm{Cen}(L)) = 2^n$. Now we shall show that the number of center elements in the closed interval $J = [x \wedge y, x \vee y \vee c(x_1)]$ is greater than 2^n. To do so, consider the following pairs of center elements in J:

$(x, y \vee c(x_1))$

$(x \vee x_2, y \vee x_3 \vee x_4 \vee \ldots \vee x_n)$, $(x \vee x_3, y \vee x_2 \vee x_4 \vee \ldots \vee x_n)$,

$\ldots$, $(x \vee x_n, y \vee x_2 \vee \ldots \vee x_{n-1})$

$(x \vee x_2 \vee x_3, y \vee x_4 \vee \ldots \vee x_n)$, $(x \vee x_2 \vee x_4, y \vee x_3 \vee x_5 \vee \ldots \vee x_n)$,

$\ldots$, $(x \vee x_{n-1} \vee x_n, y \vee x_1 \vee \ldots \vee x_{n-2})$

. . . .

$(x \vee x_3 \vee \ldots \vee x_n, y \vee x_2)$, $(x \vee x_2 \vee x_4 \vee \ldots \vee x_n, y \vee x_3)$,

$\ldots$, $(x \vee x_2 \vee \ldots \vee x_{n-1}, y \vee x_n)$

$(x \vee c(x_1), y)$.

Obviously, the number of all pairs is $\binom{n-1}{0} + \binom{n-1}{1} + \ldots + \binom{n-1}{n-1} = 2^{n-1}$. Since L is distributive and $x_i \wedge x_j = 0$ $(i \neq j)$ and $x_i \wedge y = x_i \wedge x = 0$ $(i \neq 1)$, it is not hard to see that not only are these pairs all distinct, but also all elements of these pairs are distinct from each other and any two elements in a pair are complements of each other in J. Moreover, x and y are different from both $x \wedge y$ and $x \vee y \vee c(x_1)$ since x and y are incomparable. Also, any element in a pair is different from both $x \wedge y$ and $x \vee y \vee c(x_1)$. Consequently, adjoining the pair

$(x \wedge y, x \vee y \vee c(x_1))$ of the trivial center elements in J, we have $2^n + 2$ center elements in J. Hence, we have

$$\mathrm{Card}(\mathrm{Cen}(L)) = 2^n < \mathrm{Card}(\mathrm{Cen}(J)),$$

which is what we wanted to prove.

We note that there would be at least $(2^{n+1} - 2^n - 2)$ other center elements in J because the center of J forms a Boolean lattice.

Applying the above lemma, we can easily obtain the following result of Anderson [13]:

(4.9) Corollary. H_1: L is a connected distributive topological lattice with 0 and 1,

H_2: each closed chain in L is locally convex and separable,

H_3: if $a \leq b \leq c$, then $\mathrm{Card}(\mathrm{Cen}(b \vee (c \wedge L)] \leq \mathrm{Card}(\mathrm{Cen}(a \vee L))$,

H_4: $\mathrm{Card}(\mathrm{Cen}(L)) = 2^n$,

C: L is topologically isomorphic with I^n.

Proof. Since the center C in L forms a Boolean sublattice of L, it follows from H_4 that there exist exactly n atoms $x_1, x_2, \ldots, x_n$ of C. We now show that these elements $x_1, \ldots, x_n$ are independent elements over 0 in L. Clearly, $x_i \wedge x_j = 0$ $(i \neq j)$, so that $(x_1 \vee \ldots \vee x_k) \wedge x_{k+1} = 0$, $k = 1, 2, \ldots, n-1$. Suppose that $x_i \wedge L$ is not a chain for some i. Then we can find two incomparable elements x and y in $x_i \wedge L$. By Lemma (4.8), observing that $0 \leq x \wedge y \leq x \vee y \vee c(x_i)$, we have a contradiction to H_3. Thus each $x_i \wedge L$ is a chain, so by H_2, $x_i \wedge L$ is locally convex and separable. Furthermore, $x_i \wedge L$ is connected since L is. Hence, by Theorem (4.5), L is iseomorphic with an n-cell I^n. Thus the proof is complete.

We shall now show that the number of all atoms of the center, the number of all independent elements over an element, the breadth and the codimension in a distributive topological lattice with 0 and 1 are related.

(4.10) Lemma. If L is a locally compact, chainwise connected, distributive topological lattice with 0 and 1, then $\mathrm{Ca}(L) \le \mathrm{Br}(L) \le \mathrm{In}(L) \le \mathrm{cd}(L)$ provided these numbers are finite where

Ca(L): the number of all atoms in the center in L,

Br(L): the breadth of L,

In(L): the maximal number of independent elements over an element of L, and

cd(L): the codimension of L.

Proof. (i) $\mathrm{Ca}(L) \le \mathrm{Br}(L)$. Consider the center C in L and the set of all its dual atoms $D = \{d_1, d_2, \ldots, d_k\}$, where $k = \mathrm{Ca}(L)$. It is obvious that $\mathrm{Inf}\, F \neq 0$ for any proper subset F of D. But $\mathrm{Inf}\, D = 0$. Hence $\mathrm{Br}(L) \ge k$.

(ii) $\mathrm{Br}(L) \le \mathrm{In}(L)$. Let $\mathrm{Br}(L) = m$. By definition of the breadth, there exists a finite subset F of L such that $\mathrm{Inf}\, F \neq \mathrm{Inf}\, B$ for any subset of F with $\mathrm{Card}(B) \le m-1$. So F contains a subset A with $\mathrm{Card}(A) = m$ such that $\mathrm{Inf}\, A \neq \mathrm{Inf}\, B$ for any proper subset B of A, for, otherwise, if A is an arbitrary subset of F with $\mathrm{Card}(A) = m$, then

$\text{Inf } A = \text{Inf } B$ for some $B \subset A$ with $\text{Card}(B) \leq m-1$. Therefore, since $\text{Inf}(A \cup x) = \text{Inf}(B \cup x) = \text{Inf } B'$ for any $x \in F \backslash A$ and some $B' \subset B \cup x$ with $\text{Card}(B') \leq m-1$, we have finally, by repeating this method, $\text{Inf } F = \text{Inf } C$ for some $C \subset F$ with $\text{Card}(C) \leq m-1$, which is a contradiction.

Let $A = \{a_1, a_2, \ldots, a_m\}$ be such a subset of F, and let $x_i = \text{Inf}(A \backslash a_i)$, $i = 1, 2, \ldots, m$. Also, let $a = \text{Inf } A$. Then $x_i \neq a$, $i = 1, 2, \ldots, m$ and $x_i \neq x_j$ if $i \neq j$. It is easy to see that $(x_1 \vee \ldots \vee x_k) \wedge x_k = a$, $k = 1, 2, \ldots, m-1$, i.e., $\{x_1, \ldots, x_m\}$ are independent elements over a. Hence $\text{Br}(L) \leq \text{In}(L)$.

(iii) $\text{In}(L) \leq \text{cd}(L)$. Suppose $\text{In}(L) = n$. Let $\{x_1, \ldots, x_n\}$ be a set of elements over an element a, and let $b = x_1 \vee \ldots \vee x_n$. Clearly, $M = a \vee (b \wedge L)$ is a distributive topological lattice under its relative topology. By Lemma (4.4), M is iseomorphic with $(x_1 \wedge (a \vee M)) \times \ldots \times (x_n \wedge (a \vee M))$. Since $x_i \wedge (a \vee M)$ is non-degenerate and chain-wise connected, we have $\text{cd}(x_i \wedge (a \vee M)) \geq 1$ and so $x_i \wedge (a \vee M)$ contains a non-degenerate compact connected chain. By Lemma (4.7), $\text{cd}(M) = \text{cd}((x_1 \wedge (a \vee M) \times \ldots \times x_n \wedge (a \vee M)) \geq n$. Hence, $\text{In}(L) = n \leq \text{cd}(L)$ since M is a closed subset of L.

Now we prove the following theorem:

(4.11) Theorem. If L is a locally compact chain-wise connected distributive topological lattice with 0 and 1, the following are equivalent:

(i) $Ca(L) = cd(L) = n$,

(ii) $Ca(L) = n$ and for each atom x_i of the center in L, $x_i \wedge L$ is a chain,

(iii) L is iseomorphic with a Cartesian product of n compact connected chains.

Proof. (i) → (ii): Suppose $Ca(L) = cd(L) = n$. Let $\{x_1, \ldots, x_n\}$ be the set of all atoms of the center in L. By Lemma (4.4), L is iseomorphic with $(x_1 \wedge L) \times \ldots \times (x_n \wedge L)$. Since each $x_i \wedge L$ contains a non-degenerate connected closed chain, we have $cd(x_i \wedge L) \geq 1$. If $cd(x_i \wedge L) > 1$ for some i, then by Lemma (4.7) $cd(L) = cd((x_1 \wedge L) \times \ldots \times (x_n \wedge L)) > n$ which is a contradiction. Hence, $cd(x_i \wedge L) = 1$, and since $x_i \wedge L$ is locally compact and connected, each $x_i \wedge L$ is a closed connected chain.

(ii) → (iii): Suppose that $x_i \wedge L$, $i = 1, 2, \ldots, n$, are chains. Since $x_i \wedge L$ is locally compact and connected, it follows that $x_i \wedge L$ is locally convex [1]; therefore $x_i \wedge L$ is a compact connected chain [18]. Again by Lemma (4.4), L is iseomorphic with a Cartesian product of n compact connected chains.

(iii) → (i): Suppose that L is iseomorphic with $C_1 \times \ldots \times C_n$ where C_i, $i = 1, 2, \ldots, n$, are compact connected chains with zero element 0_i and unit element 1_i. Clearly, $cd(L) = cd(C_1 \times \ldots \times C_n) = n$ [20]. It is easy to see that $(0_1, \ldots, 0_{i-1}, 1, 0_{i+1}, \ldots, 0_n)$, $i = 1, 2, \ldots, n$, are atoms of the center of $C_1 \times \ldots \times C_n$. Therefore, $Ca(L) \geq n = cd(L)$. Since $Ca(L) \leq cd(L)$, we have

$Ca(L) = cd(L) = n$. Thus, the proof is complete.

(4.12) Lemma. If L is a modular topological lattice and if $X = [u \wedge v, u]$ and $Y = [u \wedge v, v]$ are two closed intervals in L, then the topological sub-lattice generated by X and Y is iseomorphic to the Cartesian product of X and Y.

Proof. Let S be the topological sub-lattice generated by X and Y. By Theorem 7, pp. 73 in [2], S and $X \times Y$ are lattice-isomorphic under the mapping $f : X \times Y \to S$ defined by $f(x,y) = x \vee y$. Clearly, f is continuous. If we define a mapping $g : S \to X \times Y$ by $g(a) = (u \wedge a, v \wedge a)$, then g is also continuous, and moreover, $g = f^{-1}$. Hence, S and $X \times Y$ are iseomorphic.

One can easily extend Lemma (4.12) by induction as follows:

(4.13) Corollary. If $x_1, \ldots, x_n$ are independent elements over a in a modular topological lattice L, then the topological sub-lattice generated by the intervals $[a, x_k]$, $k = 1, 2, \ldots, n$ is iseomorphic with the Cartesian product of the $[a, x_k]$'s, $k = 1, 2, \ldots, n$.

(See the corollary, pp. 73 in [2].)

As an immediate consequence of Corollary (4.13), we have the following:

(4.14) Theorem. If L is a locally compact chainwise connected modular topological lattice, then $In(L) \le cd(L)$.

From the definition of the ordinal sum of two partially ordered sets [2], we are motivated to define the ordinal sum of two topological lattices L_1 with the unit 1_1 and L_2 with the zero 0_2 as follows:

The ordinal sum $L_1 \oplus L_2$ is the subset $(L_1 \times \{0_2\}) \cup (\{1_1\} \times L_2)$ of the Cartesian product $L_1 \times L_2$.

Clearly, $L_1 \oplus L_2$ is also a topological lattice under its relative topology in $L_1 \times L_2$. Moreover, the element $(1_1, 0_2)$ of $L_1 \oplus L_2$ is at the same time the greatest element of $L_1 \times \{0_2\}$ and the least element of $\{1_1\} \times L_2$.

Recall that an element a of a connected topological space L is a cut point if and only if L a is not connected [27].

The following theorem, which is due to A. D. Wallace, appears in [18].

(4.15) Theorem. If L is a connected topological lattice and if $p \in L$, then p is a cut point of L if and only if $p \neq 0$, $p \neq 1$, and $L = (p \wedge L) \cup (p \vee L)$.

(4.16) Lemma. Let L be a connected topological lattice and let $a \in L$, $a \neq 0$, and $a \neq 1$. a is a cut point of L if and only if there exists an iseomorphism $f : L \to (a \wedge L) \oplus (a \vee L)$ such that $f(a) = (a,a)$.

Proof. Suppose that a is a cut point of L. By Theorem (4.15) $L = (a \wedge L) \cup (a \vee L)$. Now we may define a mapping $f : L \to (a \wedge L) \oplus (a \vee L)$ as the following:

$$f(x) = (a,x) \quad \text{if} \quad x \in a \wedge L$$
$$f(y) = (y,a) \quad \text{if} \quad y \in a \vee L.$$

Then clearly, f is well defined, is a lattice-isomorphism, and is continuous and also $f(a) = (a,a)$. Again we define a mapping $g : (a \wedge L) \oplus (a \vee L) \to L$ as follows:

$$g(a,x) = x \vee a\ (=x) \text{ if } (a,x) \in \{a\} \times (a \vee L)$$

$$g(y,a) = a \wedge y\ (=y) \text{ if } (y,a) \in (a \wedge L) \times \{a\}.$$

It is also easy to see that g is well defined and continuous. Furthermore, $g = f^{-1}$, i.e., gf and fg are both the identity.

Conversely, suppose that L is iseomorphic with $(a \wedge L) \oplus (a \vee L)$ by a mapping f with $f(a) = (a,a)$. By definition

$$(a\wedge L)\oplus(a\vee L)=((a\wedge L)\times\{a\})\cup(\{a\}\times(a\vee L))\subset(a\wedge L)\times(a\vee L).$$

Clearly, the element (a,a) is a cut point of $(a \wedge L) \oplus (a \vee L)$, so the element a is a cut point of L since f is a lattice-isomorphism and is also homeomorphism. Thus, the proof is complete.

(4.17) Lemma. Let L be a compact connected distributive topological lattice with $cd(L) = n$. If $\mathrm{Card}(\mathrm{Cen}([a,b])) \geq \mathrm{Card}(\mathrm{Cen}([c,d]))$ for any $[c,d] \subset [a,b]$ in L, then the closed interval $[a,b]$ is iseomorphic with a Cartesian product of m compact connected chains ($m \leq n$).

Proof. Since $[a,b] = a \vee (b \wedge L)$, $[a,b]$ is also a compact connected distributive topological lattice under its relative topology in L. Let us first suppose that $\mathrm{Card}(\mathrm{Cen}([a,b])) = 2$. It follows that $[a,b]$ is a chain because otherwise, there would exist two incomparable elements x and y in $[a,b]$ so that $\mathrm{Card}(\mathrm{Cen}([x \wedge y, x \vee y]))$ would be > 2, which is a contradiction. Now suppose

$\mathrm{Card}(\mathrm{Cen}([a,b])) = 2^m$ where $m > 1$. By Theorem (4.3) we see that $m \leq n$. Let $x_1, x_2, \ldots, x_m$ be all the atoms of the center of $[a,b]$. By Lemma (4.4), $[a,b]$ is iseomorphic with $(x_1 \wedge [a,b]) \times (x_2 \wedge [a,b]) \times \ldots \times (x_m \wedge [a,b])$. From Lemma (4.8) we can easily see that each $x_i \wedge [a,b]$ is a chain. On the other hand, each $x_i \wedge [a,b]$ is compact and connected. Hence, the proof is complete.

Now we prove the following theorem:

(4.18) Theorem. Let L be a compact connected distributive topological lattice with $\mathrm{cd}(L) = n$. L is iseomorphic with $(C_1 \times C_2 \times \ldots \times C_p) \oplus (D_1 \times D_2 \times \ldots \times D_m)$ where C_i and D_j are closed connected chains, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, m$, if and only if there exists a cut point a such that $\mathrm{Card}(\mathrm{Cen}(a \wedge L)) \geq \mathrm{Card}(\mathrm{Cen}([b,c]))$ whenever $b \leq c \leq a$, and $\mathrm{Card}(\mathrm{Cen}(a \vee L)) \geq \mathrm{Card}(\mathrm{Cen}([d,e]))$ whenever $a \leq d \leq e$. Moreover, $\mathrm{cd}(L) = \mathrm{Br}(L) = \max\{p,m\}$.

Proof. Suppose that L is iseomorphic with $(C_1 \times \ldots \times C_p) \oplus (D_1 \times \ldots \times D_m)$ by a mapping f. By Lemma (4.13), $((1_1, \ldots, 1_p), (0_1, \ldots, 0_m))$ is a cut point of $(C_1 \times \ldots \times C_p) \oplus (D_1 \times \ldots \times D_m)$ where 1_i and 0_j are unit element of C_i and zero element of D_j, respectively, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, m$. So there exists a cut point a in L such that $f(a) = ((1_1, \ldots, 1_p), (0_1, \ldots, 0_m))$ and $a \wedge L$ and $a \vee L$ are iseomorphic with $C_1 \times \ldots \times C_p$ and $D_1 \times \ldots \times D_m$, respectively. Clearly, it follows that $\mathrm{Card}(\mathrm{Cen}(a \wedge L)) \geq \mathrm{Card}(\mathrm{Cen}([b,c]))$ whenever $b \leq c \leq a$ and $\mathrm{Card}(\mathrm{Cen}(a \vee L)) \geq \mathrm{Card}(\mathrm{Cen}([d,e]))$ whenever $a \leq d \leq e$.

Conversely, by Lemma (4.16), L is iseomorphic with $(a \wedge L) \oplus (a \vee L)$ and by Lemma (4.17), $a \wedge L$ and $a \vee L$ are iseomorphic with a Cartesian product $C_1 \times C_2 \times \dots \times C_p$ and $D_1 \times D_2 \times \dots \times D_m$ of compact connected chains where $p \leq n$ and $m \leq n$. Hence, L is iseomorphic with $(C_1 \times C_2 \times \dots \times C_p) \oplus (D_1 \times D_2 \times \dots \times D_m)$.

On the other hand, $cd(C_1 \times \dots \times C_p) = p$ and $cd(D_1 \times \dots \times D_m) = m$. Therefore, $cd(L) = Br(L) = \max\{p,m\}$.

One can easily extend the definition of the ordinal sum of two topological lattices to arbitrary finite numbers of topological lattices as follows:

If L_i is a topological lattice with the zero element 0_i and the unit element 1_i, $i = 1, 2, \dots, m$, then the ordinal sum of $L_1, L_2, \dots, L_m$ is defined as a subset

$$\cup_{i=1}^{m}(\{1_1\}\times\{1_2\}\times\dots\times\{1_{i-1}\}\times L_i\times\{0_{i+1}\}\times\dots\times\{0_m\})$$

of the Cartesian product $L_1 \times L_2 \times \dots \times L_m$.

Obviously, the ordinal sum operation $\oplus$ is associative up to iseomorphism but is not commutative.

One can also extend Theorem (4.18) as follows:

(4.18)' Theorem. Let L be a compact connected distributive topological lattice with $cd(L) = n$. L is iseomorphic with

$$(A_1\times A_2\times\dots\times A_p)\oplus(B_1\times B_2\times\dots\times B_q)\oplus\dots\dots\oplus(D_1 \quad ..\times D_r)$$

where $A_i, B_j, \dots, D_k$ are closed connected chains, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$, $\dots$, $k = 1, 2, \dots, r$, if and only if there exists a finite chain $0 = a_o < a_1 < \dots < a_m = 1$ of cut points $a_1, a_2, \dots, a_{m-1}$ of L such that

$\mathrm{Card}(\mathrm{Cen}([a_{i-1}, a_i])) \geq \mathrm{Card}((\mathrm{Cen}([c,d]))$ whenever $a_{i-1} \leq c \leq d \leq a_i$, $i = 1, 2, \ldots, m$. Moreover, $\mathrm{cd}(L) = \mathrm{Br}(L) = \max\{p, q, \ldots, r\}$.

The following theorem which is due to Cohen appears in [20]:

(4.19) Theorem. (Sum theorem) If X is a locally compact Hausdorff space and if X is the union of a countable family of closed sets each in $D^n(G)$ (see Section II), then $X \in D^n(G)$.

Now we shall prove the following theorem:

(4.20) Theorem. If L is a locally compact connected topological lattice and if L is the union of a countable family of compact connected chains of L, then L is a chain.

Proof. If C is a non-void, non-degenerate compact connected chain of L, then $\mathrm{cd}(C) = 1$. For, C is a compact connected topological lattice under its relative topology, and hence, C has the order topology (see [19]). It follows that $\mathrm{ind}(C) = 1$. From a result of Cohen [20] we have $\mathrm{cd}(C) = 1$. Therefore, $C \in D^1(G)$. By Sum theorem, we obtain $L \in D^1(G)$. It follows that $\mathrm{cd}(L) \leq 1$. By a result of Anderson [1], L is a chain since L is locally compact and connected.

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BIOGRAPHICAL SKETCH

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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